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REFLECTION AND REFRACTION OF A PLANAR ACOUSTIC WAVE IN AN ANISOTROPIC INHOMOGENEOUS LAYER

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The problem of reflection and refraction of a planar acoustic wave by an inhomogeneous elastic layer whose material possesses general-type anisotropy is considered. The equations of motion of the elastic layer are reduced to a system of ordinary differential equations. The boundary-value problem for this system is solved by two methods: by reduction to problems with initial conditions and by the method of power series. Analytical expressions that describe acoustic fields outside the layer are obtained. Calculation results of the transmission factor for transversely isotropic layers inhomogeneous in thickness are presented.

Reflection and transmission of acoustic waves through a planar inhomogeneous elastic layer was studied by Prikhod'ko and Tyutekin [1]. The elastic layer was assumed to be isotropic. The transmission of sound through an anisotropic layer was examined by Lonkevich [2] and Shenderov [3]. A homogeneous transversely isotropic elastic layer was considered in these papers. Skobel'tsyn and Tolokonnikov [4] considered transmission of acoustic waves through a transversely isotropic inhomogeneous elastic layer. In the present paper, the problem of transmission of a planar monochromatic acoustic wave through a planar inhomogeneous elastic layer whose material possesses general-type anisotropy is solved.

1. We consider an anisotropic inhomogeneous planar layer of thickness $2h$. The modulus of elasticity and the density of the material of this layer are described by continuous differentiable functions of the coordinate x_3. The system of rectangular coordinates x_1, x_2, x_3 is chosen so that the x_1 axis lies in the middle plane of the layer and the x_3 axis is directed downward, normal to the layer surface. We assume that the upper and lower surfaces of the layer are interfaces with ideal homogeneous liquids with densities ρ_1 and ρ_2 and speeds of sound c_1 and c_2, respectively.

Let a planar acoustic wave be incident onto an elastic layer from the half-space $x_3 < -h$. The velocity potential in this wave is

$$\psi_i = \exp\{i[k_{11}x_1 + k_{13}(x_3 + h) - \omega t]\}, \tag{1.1}$$

where $k_{11} = k_1 \sin\theta_1$ and $k_{13} = k_1 \cos\theta_1$ are the projections of the wave vector $\boldsymbol{k}_1$ onto the coordinate axes x_1 and x_3, respectively, $k_1 = \omega/c_1$ is the wavenumber in the upper half-space, ω is the angular frequency, and θ_1 is the angle of incidence of the planar wave. In what follows, we omit the time factor $\exp(-i\omega t)$.

We find the waves reflected from the layer and transmitted through the layer, and the displacement field inside the elastic layer.

2. Propagation of elastic waves in an inhomogeneous anisotropic layer is described by the general equations of an elastic medium [5]

$$\rho\ddot{u}_i = \frac{\partial\sigma_{ij}}{\partial x_j}, \qquad i = 1,\ 2,\ 3, \tag{2.1}$$

where $\rho = \rho(x_3)$ is the density of the material of the layer, u_i is the projection of the displacement vector $\boldsymbol{u}$ onto the x_i axis, and σ_{ij} are the components of the stress tensor, which are related to the components of the

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strain tensor in the general case of anisotropy by the formula

$$\sigma_{ij} = \lambda_{ijkl}\varepsilon^{kl}. \tag{2.2}$$

Here $\lambda_{ijkl} = \lambda_{ijkl}(x_3)$ are the moduli of elasticity of the anisotropic material and $\varepsilon_{kl} = (1/2)(\partial u_k/\partial x_l + \partial u_l/\partial x_k)$.

In what follows, we use two subscripts in the elasticity moduli λ_{ik}, where i, $k = 1, 2, \dots, 6$. The subscripts $1, 2, \dots, 6$ correspond to the pairs of the subscripts 11, 22, 33, 23, 13, and 12.

Since the wave vector of the incident wave lies in the plane x_1, x_3 and, hence, the exciting field is independent of the coordinate x_2 and the inhomogeneity of the material of the layer is manifested only along the x_3 axis, the coordinate x_2 should not affect the reflected field, the field transmitted into the half-space $x_3 > h$, or the field excited in the elastic layer. We also note that, in accordance with Snell's law [6], the components of the displacement vector depend on the coordinate x_1 as $\exp(ik_{11}x_1)$. Therefore, we seek the projections of the displacement vector in the form

$$u_i = U_i(x_3)\exp(ik_{11}x_1). \tag{2.3}$$

Substituting expressions (2.2) into (2.1) with account of (2.3), we obtain the following system of second-order linear ordinary differential equations for unknown functions $U_i(x_3)$ ($i = 1, 2, 3$):

$$A\boldsymbol{U}'' + B\boldsymbol{U}' + C\boldsymbol{U} = 0. \tag{2.4}$$

Here

$$\boldsymbol{U} = (U_1, U_2, U_3)^{\mathrm{t}}, \qquad A = \begin{pmatrix} \lambda_{55} & \lambda_{54} & \lambda_{53} \\ \lambda_{45} & \lambda_{44} & \lambda_{43} \\ \lambda_{35} & \lambda_{34} & \lambda_{33} \end{pmatrix},$$

$$B = \begin{pmatrix} \lambda'_{55} + 2s\lambda_{51} & \lambda'_{54} + s(\lambda_{56} + \lambda_{14}) & \lambda'_{53} + s(\lambda_{55} + \lambda_{13}) \\ \lambda'_{45} + s(\lambda_{41} + \lambda_{65}) & \lambda'_{44} + 2s\lambda_{46} & \lambda'_{43} + s(\lambda_{45} + \lambda_{63}) \\ \lambda'_{35} + s(\lambda_{31} + \lambda_{55}) & \lambda'_{34} + s(\lambda_{36} + \lambda_{54}) & \lambda'_{33} + 2s\lambda_{35} \end{pmatrix},$$

$$C = \begin{pmatrix} s\lambda'_{51} + s^2\lambda_{11} + \rho\omega^2 & s\lambda'_{56} + s^2\lambda_{16} & s\lambda'_{55} + s^2\lambda_{15} \\ s\lambda'_{41} + s^2\lambda_{61} & s\lambda'_{46} + s^2\lambda_{66} + \rho\omega^2 & s\lambda'_{45} + s^2\lambda_{65} \\ s\lambda'_{31} + s^2\lambda_{51} & s\lambda'_{36} + s^2\lambda_{56} & s\lambda'_{35} + s^2\lambda_{55} + \rho\omega^2 \end{pmatrix}$$

($s = ik_{11}$; the primes denote derivatives with respect to the coordinate x_3).

The reflected and transmitted acoustic waves are the solutions of the Helmholtz equations [6] $\Delta\psi_j + k_j^2\psi_j = 0$ ($j = 1$, 2), where $k_2 = \omega/c_2$ is the wavenumber in the lower half-space. We seek the velocity potentials of the waves reflected through the layer and transmitted through the layer in the form

$$\psi_1 = A_1 \exp\{i[k_{11}x_1 - k_{13}(x_3 + h)]\}, \qquad \psi_2 = A_2 \exp\{i[k_{21}x_1 + k_{23}(x_3 - h)]\}, \tag{2.5}$$

where k_{21} and k_{23} are the projections of the wave vector $\boldsymbol{k}_2$ onto the axes x_1 and x_3; $k_{21}^2 + k_{23}^2 = k_2^2$. According to Snell's law, we have $k_{21} = k_{11}$.

The coefficients A_1 and A_2 are to be determined from the boundary conditions, which are the equality of normal velocities of the particles of the elastic medium and the liquid on both surfaces of the planar layer, the absence of shear stresses on these surfaces, and the equality of normal stress and acoustic pressure on them:

$$\begin{aligned} -i\omega u_3 = v_{1n}, \quad \sigma_{13} = 0, \quad \sigma_{23} = 0, \quad \sigma_{33} = -p_1 \quad \text{for} \quad x_3 = -h, \\ -i\omega u_3 = v_{2n}, \quad \sigma_{13} = 0, \quad \sigma_{23} = 0, \quad \sigma_{33} = -p_2 \quad \text{for} \quad x_3 = h. \end{aligned} \tag{2.6}$$

Here $v_{1n} = \partial(\psi_i + \psi_1)/\partial x_3$, $p_1 = i\omega\rho_1(\psi_i + \psi_1)$ and $v_{2n} = \partial\psi_2/\partial x_3$, $p_2 = i\omega\rho_2\psi_2$ are the normal velocity components of the liquid particles and the acoustic pressures in the upper and lower half-spaces, respectively.

We substitute expressions (1.1), (2.2), (2.3), and (2.5) into boundary conditions (2.6). As a result, we obtain the expressions

$$A_1 = 1 + (\omega/k_{13})U_3(-h), \qquad A_2 = -(\omega/k_{23})U_3(h) \tag{2.7}$$

for the coefficients A_1 and A_2 and six conditions for finding the partial solution of the system of differential equations (2.4):

$$(A\boldsymbol{U}' + E\boldsymbol{U})_{x_3=-h} = \boldsymbol{D}; \tag{2.8}$$

$$(A\boldsymbol{U}' + F\boldsymbol{U})_{x_3=h} = 0, \tag{2.9}$$

Here

$$E = s\begin{pmatrix} \lambda_{51} & \lambda_{56} & \lambda_{55} \\ \lambda_{41} & \lambda_{46} & \lambda_{45} \\ \lambda_{31} & \lambda_{36} & \lambda_{35} + \dfrac{i\rho_1\omega^2}{sk_{13}} \end{pmatrix}, \qquad F = s\begin{pmatrix} \lambda_{51} & \lambda_{56} & \lambda_{55} \\ \lambda_{41} & \lambda_{46} & \lambda_{45} \\ \lambda_{31} & \lambda_{36} & \lambda_{35} - \dfrac{i\rho_2\omega^2}{sk_{23}} \end{pmatrix},$$

$$\boldsymbol{D} = (0; 0; -2i\rho_1\omega)^{\mathrm{t}}.$$

It follows from formulas (2.7) that the reflection factor A_1 and the transmission factor A_2 can be calculated only after determining the values of the function $U_3(x_3)$ on the surfaces of the layer.

3. To find the field of displacements in an elastic layer, we have to solve boundary-value problem (2.4), (2.8), (2.9). We reduce this problem to the problem with initial conditions. Let $\boldsymbol{U}_1, \boldsymbol{U}_2, \ldots, \boldsymbol{U}_6$ form a fundamental system of the solutions of Eqs. (2.4) on the interval $(-h, h)$. In this case, the solution $\boldsymbol{U}$ of the boundary-value problem is an arbitrary linear combination

$$\boldsymbol{U} = \sum_{j=1}^{6} C_j \boldsymbol{U}_j. \tag{3.1}$$

Substituting (3.1) into boundary conditions (2.8) and (2.9), we obtain the following system of six linear algebraic equations for the unknown coefficients C_j ($j = 1, 2, \ldots, 6$):

$$\sum_{j=1}^{6} C_j(A\boldsymbol{U}_j' + E\boldsymbol{U}_j)_{x_3=-h} = \boldsymbol{D}, \qquad \sum_{j=1}^{6} C_j(A\boldsymbol{U}_j' + F\boldsymbol{U}_j)_{x_3=h} = 0. \tag{3.2}$$

Determining the coefficients $C_1, C_2, \ldots, C_6$, we find the functions U_1, U_2, and U_3 from formula (3.1). The components of the displacement vector are now determined by expressions (2.3), and the acoustic fields outside the elastic layer are found from (2.5) and (2.7).

We consider the order of constructing the fundamental system of the solutions for Eqs. (2.4). The existence condition for a fundamental system of solutions for Eqs. (2.4) determined and continuous on the interval $(-h, h)$ is the continuity of the coefficients of system (2.4) on this interval. This condition is satisfied if not only the functions $\rho(x_3)$ and $\lambda_{ik}(x_3)$, but also the first derivatives $\lambda_{ik}'(x_3)$, are continuous. In addition, $\det A$ should be other than zero on this interval. As a fundamental system of the solutions, we can choose six arbitrary solutions of the Cauchy problem for system (2.4) with initial conditions, which are linearly independent. We can choose the initial conditions

$$\boldsymbol{U}_j = (\delta_{1j}, \delta_{2j}, \delta_{3j})^{\mathrm{t}}, \qquad \boldsymbol{U}_j' = (\delta_{4j}, \delta_{5j}, \delta_{6j})^{\mathrm{t}} \qquad (j = 1, 2, \ldots, 6), \tag{3.3}$$

where j is the ordinal number of the Cauchy problem and δ_{ij} is the Kronecker delta. The initial point can be any point within $[-h, h]$. The Cauchy problems can be solved by some numerical method.

4. We construct an approximate analytical solution of boundary-value problem (2.4), (2.8), (2.9). We use the method of power series [7]. In this case, it is required that the function $\rho(x_3)$ and its first derivative be continuous on the interval $[-h, h]$, and the functions $\lambda_{ik}(x_3)$ be continuous and have continuous derivatives up to the second order inclusive and $\det A \neq 0$.

We write the boundary-value problem in the dimensionless variables $x = x_3/h$, $\boldsymbol{U}_j^* = \boldsymbol{U}_j/h$, $\lambda_{ik}^* = \lambda_{ik}/\lambda_0$, and $\rho^* = \rho/\rho_0$, where λ_0 and ρ_0 are certain characteristic elastic constant and density. In what follows, we come back to four subscripts for designation of the moduli of elasticity, which allows us to write the derived relationships in a more compact form. We assume that the moduli of elasticity and the density of the material of the layer have the form of polynomials relative to x (or they are approximated by these polynomials):

$$\lambda_{ijkl}^*(x) = \sum_{m=0}^{R} \lambda_{ijkl}^{(m)} x^m, \qquad \rho^*(x) = \sum_{m=0}^{R} \rho^{(m)} x^m. \tag{4.1}$$

Here $\lambda_{ijkl}^{(m)}$ and $\rho^{(m)}$ are the coefficients of the polynomials and R is the maximum power of the polynomials used.

Taking into account the above constraints, we can seek the solution of system (2.4) in the form

$$U_i^* = \sum_{n=0}^{\infty} U_i^{(n)} x^n \qquad (i = 1,\ 2,\ 3), \tag{4.2}$$

and the series (4.2) converges on the interval $[-1, 1]$.

We obtain recurrent relations for finding the coefficients $U_i^{(n)}$. We write system (2.4) in a coordinate (dimensionless) form

$$\sum_{i=1}^{3} (A_{ki}^* U_i^{*''} + B_{ki}^* U_i^{*'} + C_{ki}^* U_i^*) = 0 \qquad (k = 1,\ 2,\ 3). \tag{4.3}$$

On the basis of expressions (4.1), we write the dimensionless elements of the matrices A^*, B^*, and C^* in the form of the polynomials

$$A_{ki}^* = \sum_{m=0}^{R} A_{ki}^{(m)} x^m, \qquad B_{ki}^* = \sum_{m=0}^{R} B_{ki}^{(m)} x^m, \qquad C_{ki}^* = \sum_{m=0}^{R} C_{ki}^{(m)} x^m, \tag{4.4}$$

where

$$A_{ki}^{(m)} = \lambda_{k3i3}^{(m)}, \qquad B_{ki}^{(m)} = sh(\lambda_{k3i1}^{(m)} + \lambda_{k1i3}^{(m)}) + (m+1)\lambda_{k3i3}^{(m+1)},$$

$$C_{ki}^{(m)} = sh\Big[(m+1)\lambda_{k3i1}^{(m+1)} + sh\lambda_{k1i1}^{(m)} + \frac{\omega^2 h}{s\lambda_0} \rho_0 \rho^{(m)} \delta_{ki}\Big].$$

Note that $\lambda_{ijkl}^{(m)} = 0$ and $\rho^{(m)} = 0$ for $m > R$.

Substituting expressions (4.2) and (4.4) into Eq. (4.3) and equating to zero the coefficients at different powers of x, we obtain equations for determining the coefficients $U_i^{(n)}$. Resolving the latter with respect to $U_i^{(n+2)}$, we find

$$\boldsymbol{U}^{(n+2)} = -\frac{A^{(0)-1}}{(n+1)(n+2)} \sum_{m=0}^{R_1} [G^{(m)} \boldsymbol{U}^{(n+1-m)} + C^{(m)} \boldsymbol{U}^{(n-m)}], \tag{4.5}$$

where $\boldsymbol{U}^{(n)} = (U_1^{(n)}, U_2^{(n)}, U_3^{(n)})^{\mathrm{t}}$, $G^{(m)} = (G_{ki}^{(m)})_{3\times 3}$, $C^{(m)} = (C_{ki}^{(m)})_{3\times 3}$, $G_{ki}^{(m)} = (n+1-m)\times [(n+1)\lambda_{k3i3}^{(m+1)} + sh\,(\lambda_{k3i1}^{(m)} + \lambda_{k1i3}^{(m)})]$, and $R_1 = \min(R, n)$.

Recurrent relation (4.5) allows us to calculate all the coefficients of expansions (4.2) except for $U_i^{(0)}$ and $U_i^{(1)}$ ($i = 1$, 2, 3). The coefficients $U_i^{(0)}$ and $U_i^{(1)}$ can be easily determined if we use the reduction of the boundary-value problem to the Cauchy problems with initial conditions (3.3) at the point $x = 0$. We obtain the following solution (for the Cauchy problem numbered j):

$$\boldsymbol{U}_j^{(0)} = (\delta_{1j}, \delta_{2j}, \delta_{3j})^{\mathrm{t}}, \qquad \boldsymbol{U}_j^{(1)} = (\delta_{4j}, \delta_{5j}, \delta_{6j})^{\mathrm{t}} \qquad (j = 1,\ 2, \ldots, 6). \tag{4.6}$$

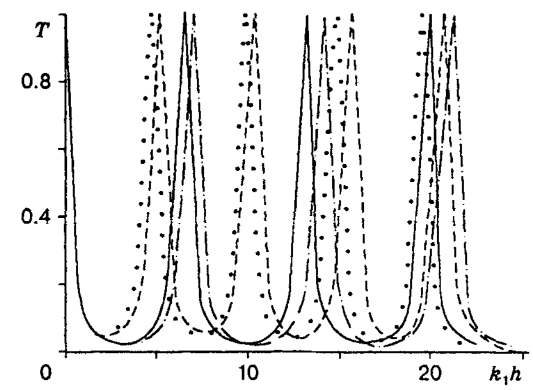

Fig. 1

The solution of system (4.3) is

$$U_i^* = \sum_{j=1}^{6} C_j U_{ij}^*, \tag{4.7}$$

where $U_{ij}^* = \sum_{n=0}^{\infty} U_{ij}^{(n)} x^n$ ($i = 1$, 2, and 3 and $j = 1, 2, \ldots, 6$). The coefficients $U_{ij}^{(n)}$ are calculated from formulas (4.5) and (4.6) for $j = 1, 2, \ldots, 6$. The coefficients C_j are found from the boundary conditions by solving the system of linear algebraic equations (3.2) written in dimensionless quantities. After finding the coefficient C_j, we obtain an approximate analytical solution of boundary-value problem (2.4), (2.8), (2.9) in the form (4.7).

5. Based on the solution obtained, we calculated the transmission factor from the intensity $T = (\rho_2 c_1/(\rho_1 c_2))|A_2|^2$ for a transversely isotropic plate located in water ($\rho_1 = \rho_2 = 10^3$ kg/m^3 and $c_1 = c_2 = 1485$ m/sec). Numerical studies were performed both for homogeneous materials with density $\rho_0 = 2.7 \cdot 10^3$ kg/m^3, and for a layer with the following type of inhomogeneity: $\rho(x_3) = \rho_0 f(x_3)$, where $f(x_3) = a\{0.2 + \exp[-16(x_3 + h)^2]\}$. The factor a was chosen such that the mean value of the function $f(x_3)$ over the layer thickness was equal to unity. In our calculations, we used the following values of the moduli of elasticity of the anisotropic material: $\lambda_{11} = 16.4 \cdot 10^{10}$ N/m^2, $\lambda_{13} = 3.28 \cdot 10^{10}$ N/m^2, $\lambda_{33} = 5.74 \cdot 10^{10}$ N/m^2, and $\lambda_{44} = 2.54 \cdot 10^{10}$ N/m^2. The value of the modulus of elasticity λ_{12} is not fixed, since it enters neither Eqs. (2.4) nor boundary conditions (2.8) and (2.9) in the case of a transversely isotropic layer. To estimate the effect of anisotropy of the material of the layer on the transmission of sound, we also performed calculations for an isotropic layer whose material had the same density as the anisotropic layer and occupied an intermediate position in terms of the velocity of longitudinal waves relative to the velocity of quasi-longitudinal waves in the anisotropic material considered. For the chosen isotropic material, we had $\lambda_{11} = 10.5 \cdot 10^{10}$ N/m^2, $\lambda_{13} = 5.3 \cdot 10^{10}$ N/m^2, $\lambda_{33} = 10.5 \cdot 10^{10}$ N/m^2, and $\lambda_{44} = 2.6 \cdot 10^{10}$ N/m^2.

In numerical studies, boundary-value problem (2.4), (2.8), (2.9) was solved by two methods: the fourth-order Runge–Kutta scheme with an automatic choice of the integration step and the method of power series. The calculation results obtained by the two methods were in good agreement.

Figure 1 shows the frequency dependence of the transmission factor T for the case of normal incidence of an acoustic wave on the layer (the solid curve corresponds to a homogeneous isotropic layer, the dashed curve refers to an inhomogeneous anisotropic layer, the dot-and-dash curve to an inhomogeneous isotropic layer, and the dotted curve to a homogeneous anisotropic layer). At resonance frequencies, the elastic layer is fully transparent for an incident acoustic wave. In the range of low frequencies ($k_1 h < 1$), the transmission of sound is affected by neither anisotropy nor inhomogeneity of the material.

The graphs plotted for homogeneous layers exhibit a clear periodicity of the appearance of resonances with a period $k_1\pi/k_l$, where k_l is the wavenumber of longitudinal waves in the layer. In the case considered, we have $k_l = \omega\sqrt{\rho_0/\lambda_{33}}$. Hence, the relative position and the periods of appearance of resonances of homogeneous materials are determined by the value of the modulus of elasticity λ_{33}.

The inhomogeneity of the layer of the considered type shift the resonances to the range of higher frequencies as compared to the corresponding homogeneous layer. This could be expected, since the mean velocity of longitudinal waves across an inhomogeneous layer is greater than the velocity of longitudinal waves in a homogeneous layer. In addition, another special feature of frequency characteristics is manifested in inhomogeneous materials, namely, the absence of periodicity in the appearance of resonances.

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